Inelastic Column Theories and an Analysis of Experimental Observations*

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SUMMARY

The inelastic column buckling theories as usually presented are found to be rather confusing. Many experimental results still cannot be explained satisfactorily by the theories as they stand. It appears to be pertinent, therefore, to define the inelastic buckling problem once more from a more rigorous mathematical point of view and to give a more rigorous mathematical treatment of the problem. The effect of assuming a constant tangent modulus on the buckling load in the tangent modulus formula is discussed, and many anomalies of column behavior in the inelastic region are explained.

In making such a study, it is important to emphasize the difference between an ideal column and an actual column and the difference between the buckling load and the ultimate load. Southwell's method[1] for analyzing column tests, which was originally proposed for the case of elastic buckling, is now shown to be valid for inelastic buckling. It is also shown that, in applying the method, instead of analyzing load and deflection measurements, simultaneous load and strain readings can be used. Thus, it is easier to measure the strain more accurately.

INTRODUCTION

Much interest in inelastic column theory has recently been aroused by the presentation of Shanley's paper.[2] The value of the paper lies in the explanation of certain physical facts concerning the inelastic buckling process. By means of an idealized hypothetical column, Shanley has shown that "bending begins at the tangent modulus load and the column load increases with increasing lateral deflection, approaching the reduced-modulus load as a limit if the tangent modulus is assumed to remain constant." There are many ambiguities in this statement; for example, what is meant by the phrase "bending begins" and what is meant by "the column load?" And, still further, would such a conclusion, drawn from the study of an idealized hypothetical column, also be true for any other columns?

A study of the experimental results on short columns—e.g., Fig. 8—immediately reveals the previous conclusion is not quite true, because, when the column compressive stress, P/A, is in the region of the proportional limit of the material, the test results indicate that the so-called column loads are usually less than the tangent modulus load, while Shanley has shown that they must always be greater than, or equal to, the tangent modulus load. The experimental results also indicate that there is a consistent tendency for the column load to be close to the reduced or double modulus load rather than tangent modulus load when P/A is in the region of yield stress. There must be a satisfactory explanation for such consistent tendency, but it cannot be found in the present-day theories.

It is the purpose of this paper to give a more rigorous study of the inelastic column buckling theories and the related experimental observations. By a more rigorous mathematical treatment, it is found that many anomalies of column behavior observed in the laboratory can be satisfactorily explained.

ELASTIC BUCKLING THEORY AND THE DIFFERENCE BETWEEN THE BUCKLING LOAD AND THE ULTIMATE LOAD OF AN IDEAL SLENDER COLUMN

In order to clarify the discussions on inelastic column buckling, theories concerning elastic buckling will be briefly reviewed.

The problem of elastic instability consists of determining the smallest load at which the original straight form of equilibrium of a centrally loaded ideal column becomes unstable. To formulate the problem mathematically, it is usual to apply an infinitesimally small disturbance to the originally straight column and to investigate whether this bent form of equilibrium can be maintained by the axial load, P, acting alone when the disturbance is removed. The idea of applying and removing the small disturbance, though of no importance here, is important in the discussion of inelastic buckling load and will be elaborated to some extent later. Referring to Fig. 1, if such a bent form of equilibrium is possible, at any cross section perpendicular to the x-axis, the internal forces over the cross section can be reduced to a compressive force applied at the centroid of the cross section and a couple. The equilibrium equations are, therefore, as follows:

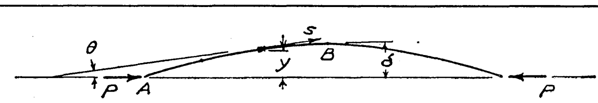

FIG. 1. The bent form of equilibrium of an ideal column under the action of axial load P.

Received August 9, 1947. Revised and received November 6, 1947.

* The author is indebted to Prof. J. S. Newell, of Massachusetts Institute of Technology; George Gerard, of New York University; and Paul Sandorff, of the Lockheed Aircraft Corporation, for their kind criticisms. To Prof. F. K. Teichmann, of New York University, the author wishes to express his indebtedness for his kind interest.

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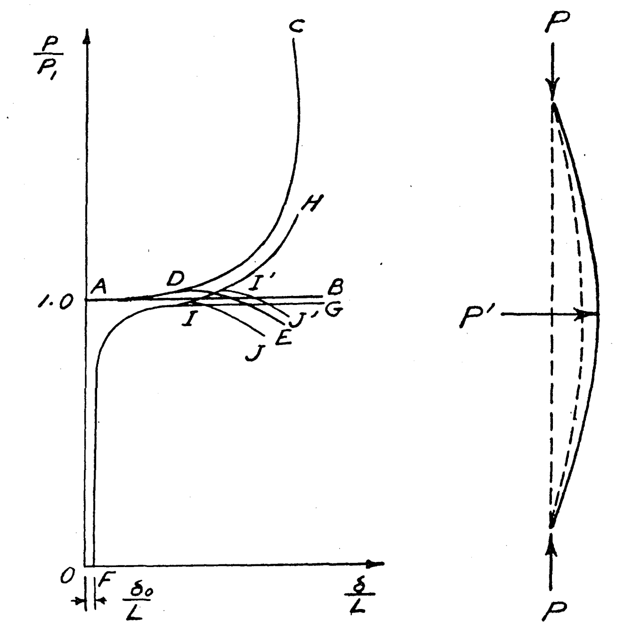

FIG. 2 (*left*). Theoretical load-deflection curves for ideal and actual slender columns. FIG. 3 (*right*). Bending of an ideal column under a lateral disturbance.

$$\sum F_x = 0 \quad \text{or} \quad \int_A \sigma_x dA = P \tag{1}$$

and

$$\sum M = 0 \quad \text{or} \quad \int_A \sigma_x \eta dA = -Py \tag{2}$$

where σ_x is the normal stress acting at an arbitrary point of the cross section which is at a distance η from the centroidal axis, A is the cross-sectional area, y is the deflection of the section from the centroidal axis, and the integration is taken over the whole cross-sectional area. Assuming that a plane perpendicular to the centroidal axis of the column remains plane after bending, the x-component of the strain, ϵ_x, at an arbitrary point in a certain section is given by

$$\epsilon_x = \epsilon_0 - (\eta/R) \tag{3}$$

where ϵ_0 is the strain at the centroidal axis, R is the radius of curvature, and

$$\frac{1}{R} = \frac{d\theta}{ds} = \frac{d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}} \tag{4}$$

where s is the distance along the centroidal axis and θ is the angle between s and x-axis. Integrating Eqs. (1) and (2) and denoting $E\epsilon_0$ by σ_0, since $\sigma_x = E\epsilon_x$ (E being the Young's modulus), it is found from Eq. (1) that $\sigma_0 = P/A$, and Eq. (2) becomes

$$(d\theta/ds) + (P/EI)y = 0 \tag{5}$$

where I is the moment of inertia of the cross-sectional area about the centroidal axis. The solution of Eq. (5) can be expressed in terms of an elliptical integral and is given in any standard treatise on the Theory of Elasticity. When the deflection is small, the curvature can be approximated by d^2y/dx^2, and Eq. (5) is reduced to the well-known Euler's equation as follows:

$$(d^2y/dx^2) + (P/EI)y = 0 \tag{6}$$

It is well known that the differential Eq. (6) will have a nontrivial solution only when

$$P_n = Cn\pi^2EI/L^2, \qquad n = 1, 2, 3, \text{etc.} \tag{7}$$

The smallest value of P is at $n = 1$ and is equal to

$$P_{cr} = C\pi^2EI/L^2 \tag{8}$$

where P_{cr} is the so-called buckling load, L is the length of the column, and C is the coefficient of end fixity. Physically, it indicates that a bent form of equilibrium is possible when the axial load is of the magnitude given by Eq. (8). Solution of Eq. (5) reveals something more. It can be shown[6] that, at and below the buckling load, the column has only one form of equilibrium—i.e., the straight form. When the load is greater than the buckling load, there are two possible forms of equilibrium—the straight form which is unstable and the bent form which is stable.

The results from Eqs. (5) and (6) are plotted in Fig. 2. Suppose that the column is slender and that the material can sustain the compressive stress $P/A = \sigma_0$ without failure of elasticity. The loading of the ideal column will follow the curve OAB according to Eq. (6) and OAC according to the more exact Eq. (5). At some point D on the curve OAC, the center deflection δ is so large that the sum of direct compressive stress and bending stress due to the bending moment $P\delta$ is sufficient to produce elastic breakdown. Beyond this point the actual curve $OADE$ will begin to drop away from OAC, since the material can now sustain less load than before. The ultimate or maximum load a column can carry then occurs at point D, and the buckling load is at A. It is thus seen that for an ideal slender column the ultimate load is always greater than the buckling load. But aside from this fact, they are entirely different theoretically, because the ultimate load is determined mainly by the mechanical properties of the material of the column, while the buckling load is determined mainly by the geometrical configuration of the column. The former is a load due to the material failure, and the latter is due to the instability of the straight equilibrium form.

LOADING OF AN ACTUAL COLUMN

Whereas an ideal column must be perfectly straight and made of material of homogeneous composition, an actual column is more or less imperfect in that it may be initially bent and may not be completely homogeneous. It is obvious that the effect due to nonhomogeneity of the material cannot be taken into account in the general theoretical discussion and will not be considered in this paper. Assume that the column is not quite straight initially. Let y_0 denote the initial deflection of the column axis from the line of thrust. Then Eq. (6) is replaced by

$$[(d^2y/dx^2) - (d^2y_0/dx^2)] + (P/EI)y = 0 \qquad (9)$$

The form of y will now depend upon the form of y_0, both quantities being regarded as functions of x. Provided that y_0 vanishes at either end of the column—i.e., although the column has initial curvature it is centroidally loaded—a general solution of Eq. (9) may be obtained by expressing y and y_0 in terms of Fourier series. Thus, if

$$y = \sum_{n=1}^{\infty} \delta_n \sin(n\pi x/L) \qquad (10)$$

$$y_0 = \sum_{n=1}^{\infty} \bar{\delta}_n \sin(n\pi x/L)$$

where δ_n and $\bar{\delta}_n$ are constants, we find on substitution that

$$\delta_n = \bar{\delta}_n/[1 - (P/P_n)], \qquad n = 1, 2, 3, \text{etc.} \quad (11)$$

The deflection of the column at its center can be obtained by substituting $x = L/2$ in Eq. (10), or

$$\delta' = \delta_1 - \delta_3 + \delta_5 - \ldots \qquad (12)$$

For an actual column corresponding to the ideal one discussed previously, the deflection versus load curve is also shown in Fig. 2. The loading of the column will follow the curve FG according to Eq. (12). However, when the deflection is large, the simplified form of curvature is no longer a good approximation. If the more exact formula for the curvature is used, the loading of the column will follow the curve FIH. At some point I or I' on the curve, the maximum stress in the column reaches the elastic limit, and then the actual curve FIJ of $FI'J'$ will drop away from FH.

It is to be noted that, for an actual column, there is only one form of equilibrium—i.e., the bent form which is stable. Therefore, there is no such equilibrium load at which an exchange of stabilities occurs. However, the curve FG does approach the line AB as an asymptote. But the loading curve will break away from the curve FG before the deflection becomes too large. The ultimate load, which corresponds to the point I or I' in Fig. 2, can be either greater or smaller than the buckling load for the corresponding ideal column. Only in the case of long columns can one expect that the ultimate load is near to the buckling load. This is because the buckling stress, P_{cr}/A, in this case is well below the proportional limit and the deflection must be considerable when the elastic limit is reached. Because of this, the maximum for P will occur somewhere on the flat part of the curve thereby close to the horizontal line AB. This agreement is a somewhat fortuitous occurrence and cannot be regarded as a general rule for shorter columns.

Inelastic Buckling Theories

When $P/A = \sigma_0$ is beyond the elastic limit, the buckling load can be defined as the smallest axial load at which the bent form of equilibrium of an originally straight and centrally loaded column, resulting from the action of a small disturbance, becomes stable. The definition, that it is the smallest load at which the straight form of equilibrium of a centrally loaded ideal column becomes unstable, is not quite adequate. This can be seen from the fact that the column, once it is bent by a disturbance, will not return to its original straight form because of permanent set, even though the disturbance is removed. In other words, the straight form may be called unstable in such cases. In the elastic region, these two definitions are the same.

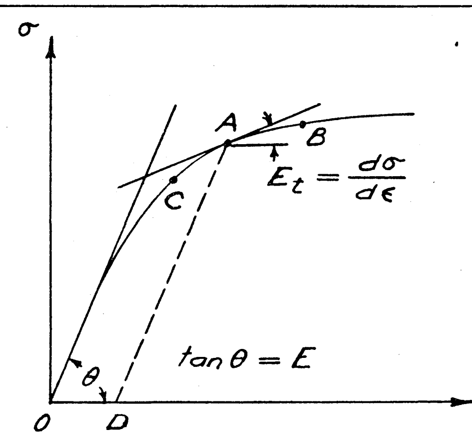

Fig. 4. A typical compressive stress-strain curve.

As the understanding of inelastic buckling theories depends greatly on the idea of the application of an infinitesimally small disturbance, a word about such disturbance appears to be pertinent. It is to be noted that a disturbance is a small force or moment that can be applied and removed at will. Referring to Fig. 3, the disturbance is represented in the form of a small lateral load P'. If P' is applied before P reaches the buckling load and is then removed, the ideal column will resume its original straight form when no fiber is stressed beyond the proportional limit. When some or all of the fibers are stressed beyond the proportional limit, the column, due to permanent set, will assume the bent form as indicated by the dotted line in the figure, which is different from the original bent form when P' is acting on the column. This indicates that the original bent form is not a stable form of equilibrium. If P' is applied to the column when P is equal to the buckling load and then P' is removed, the bent form will remain unchanged—i.e., this bent form is stable.

While there is no importance in specifying *when* the disturbance is applied and *when* it is removed in the elastic case, the magnitude of the inelastic buckling load depends intimately on such actions, for the process of loading and unloading is now irreversible. The final stress distribution at a cross section of the column can be the result of many different types of loading, depending on when the small disturbance is applied and when it is removed. The two extreme cases are as

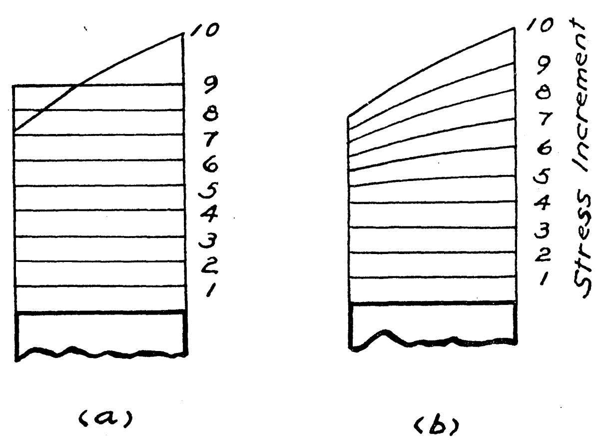

FIG. 5. Two extreme types of strain distribution across column cross section.

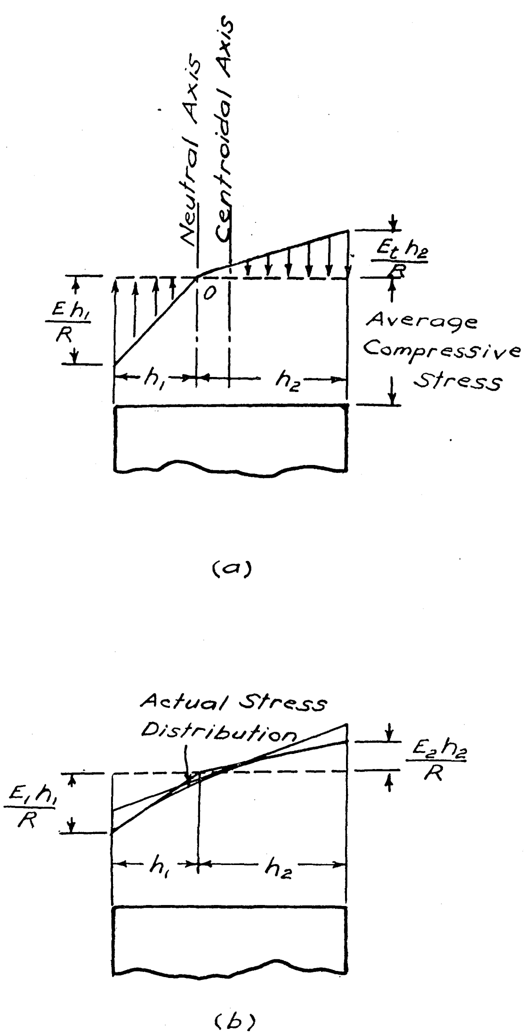

FIG. 6.

follows: First, the column may be compressed to the critical stress, say, the point A on the stress-strain diagram (Fig. 4), and then the disturbance is applied and immediately removed. As a result, the column suddenly bends. Because of the bending, the stress on the compression side is in the process of loading, following the curve AB, and the stress on the tension side is in the process of unloading, following the curve AD. The stress distribution in this case is shown in Fig. 5a. On the other hand, the disturbance may be applied during the compression process before the compressive stress reaches the proportional limit and is removed when the load is increased to a value such that the bent form becomes stable. In this case, compression and bending proceed simultaneously during the loading process, and there is no reversal of stress. The final stress distribution is shown in Fig. 5b.

Denote $E_t = d\sigma/d\epsilon$ the tangent modulus and $\sigma_0 = P/A$ the stress at the neutral axis 0, Fig. 6a. According to the first process of loading, von Kármán's double modulus formula can be derived. The derivation is well known but is briefly outlined here for the purpose of discussion.

Assuming that plane cross sections of the column remain plane during bending, the small bending stresses will be distributed along the depth of the cross section as shown in Fig. 6a. Since the bending is only slight in determining the buckling load, the stress curve on the loading side may be approximated by a straight line with a slope equal to E_t. Eq. (1) becomes

$$E \int_0^{h_1} \eta dA + E_t \int_{h_2}^{0} \eta dA = 0 \tag{13}$$

and the bending moment is

$$M = \int_A \sigma_x \eta dA = (E/R) \int_0^{h_1} \eta^2 dA + (E_t/R) \int_{h_2}^{0} \eta^2 dA = (1/R)(EI_1 + E_t I_2) \tag{14}$$

in which I_1 and I_2 are the moments of inertia with respect to the neutral axis of the two portions of the cross section. $(EI_1 + E_tI_2)$ is usually written as E_rI, where $E_r = (1/I)(EI_1 + E_tI_2)$ is the so-called von Kármán's double modulus. However, it is obvious that the effect can be better described by writing $(EI_1 + E_tI_2) = \overline{EI}$ instead of E_rI, since actually both E and I have been modified instead of changes in E alone. Since the bending is only slight, $\overline{EI}$ is approximately a constant throughout the length of the column. Eq. (2) then becomes

$$(d^2y/dx^2) + (P/\overline{EI})y = 0 \tag{15}$$

and the buckling load is

$$P_{cr} = C\pi^2\overline{EI}/L^2 \tag{16}$$

This is essentially von Kármán's double modulus formula except that $\overline{EI}$ is now written in the place of E_rI. In the case of colums with rectangular sections, it can be proved that (Fig. 6a),

$$\overline{EI} = [4EE_t/(\sqrt{E} + \sqrt{E_t})^2]I \tag{17}$$

Now let us consider the second process of loading. Let us again assume that the stress distribution over the section can be approximated by a straight line, in other words, we assume the tangent modulus is approximately constant over the stress range and it can be

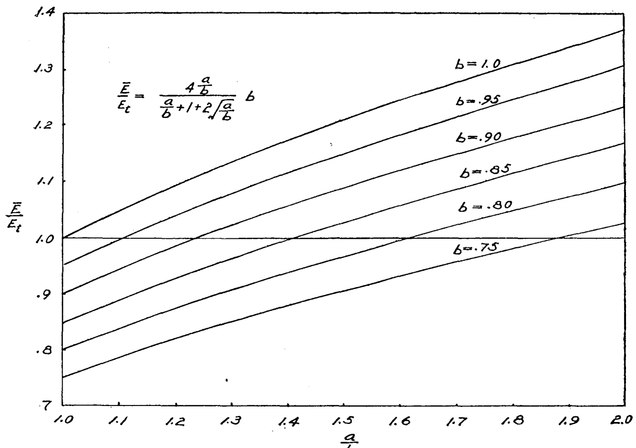

FIG. 7.

easily proved that the buckling load is given by the so-called tangent modulus formula as follows:

$$P_{cr} = C\pi^2 E_t I/L^2 \tag{18}$$

where E_t is the tangent modulus corresponding to the stress P_{cr}/A. The assumption of constant tangent modulus is usually justified by arguing that the bending is only slight; however, it would be interesting to see the influence of such an approximation on the magnitude of the buckling load. A better approximation can be obtained by approximating the stress distribution curve over the cross section by two straight lines instead of the usual one straight line, as shown in Fig. 6b. Denoting E_1 and E_2 the tangent modulus corresponding to the stresses at two sides of the neutral axis, the same results can be obtained as in the case of double modulus theory, merely by replacing E and E_t by E_1 and E_2, respectively. In the general case, the buckling load is equal to

$$P_{cr} = C\pi^2 \overline{EI}/L^2 \tag{19}$$

in which $\overline{EI} = (E_1I_1 + E_2I_2)$. For columns with rectangular sections,

$$\overline{EI} = [4E_1E_2/(\sqrt{E_1} + \sqrt{E_2})^2]I \tag{20}$$

Write $E_1 = aE_t$, $E_2 = bE_t$, where E_t as before is the tangent modulus at $\sigma_0 = P_{cr}/A$. For most engineering materials, $a \geqslant 1$ and $b \leqslant 1$, and Eq. (20) can be written as

$$\overline{EI} = [4ab/(\sqrt{a} + \sqrt{b})^2]E_tI = KE_tI \tag{21}$$

The factor

$$K = \frac{4ab}{(\sqrt{a} + \sqrt{b})^2} = \frac{4(a/b)}{(a/b) + 1 + 2\sqrt{a/b}}\,b$$

is plotted against a/b for different values of b in Fig. 7. It is seen that, for low values of b which lie in the region of the "knee" of the stress-strain curve or near the proportional limit, the factor is likely to be less than one—i.e., the tangent modulus load is greater than the actual buckling load. For higher values of b which are in the region of large stress and strain or near the yield point, the factor is likely to be greater than one. The tangent modulus load is, therefore, less than the actual buckling load. This explains why the tangent modulus

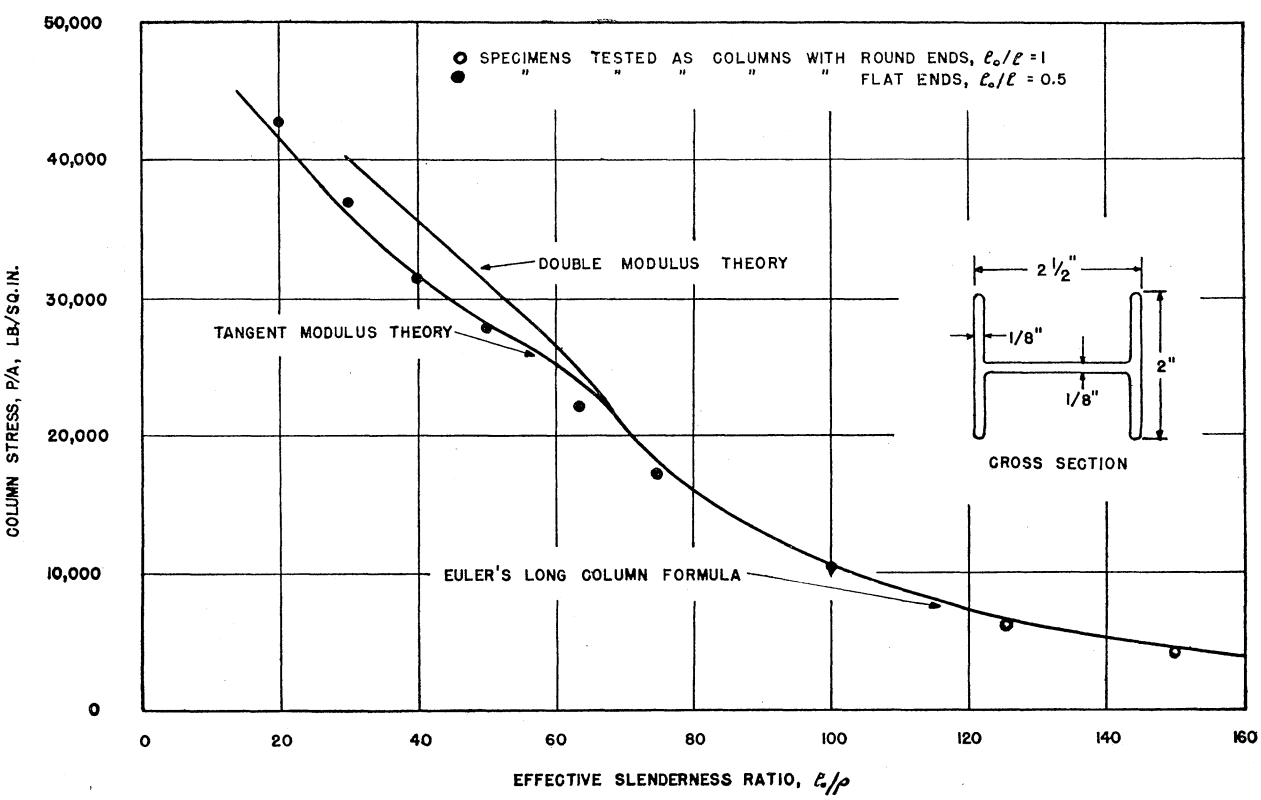

FIG. 8. A typical column curve (cf., *N.A.C.A. T.R. No. 656*).

theory predicts unconservative values when the buckling stress is near the knee of the stress-strain curves. It also is one reason why the tangent modulus theory predicts values that are too low when the buckling stress is high (Fig. 8).

Theoretically, for a centrally loaded ideal column, if a small disturbance is applied and immediately removed, the resulting bent form becomes stable when P is equal to the double modulus load, P_d. Referring to Fig. 3, in this case, P' is assumed to be applied and removed when P is equal to P_d. Whereas, if during the loading P' is applied before P/A reaches the proportional limit and is removed when P is equal to P_{cr}—i.e., load at which the bent form becomes stable—the buckling load, P_{cr}, is close to the tangent modulus load, P_t. If the disturbance, P', is applied after P/A is beyond the proportional limit but below P_d/A, then the buckling load would be equal to some value between P_d and P_t.

Since the tangent modulus, on which the values of $\overline{EI}$ depend, decreases rather rapidly shortly after the proportional limit is exceeded for most engineering materials, the effect of plastic deformation on the load is larger than the effect due to the use of approximate expression for the curvatures. The load-deflection curve would, therefore, drop immediately after buckling starts. It is thus seen that, for an ideal short column, the buckling load is equal to, or very close to, the ultimate load.

If the testing of an ideal column is possible, there will always be disturbances in the testing machine during the loading process, and the double modulus load can never be obtained. As the magnitude of the disturbance created in a given testing machine is more or less constant, its effect on bending of columns depends approximately upon the slenderness ratio L/ρ. For short columns with small L/ρ ratio, the column is relatively insensitive to small disturbances. In such cases, the column may bend when P/A is beyond the proportional limit and there is actually a reversal of stresses. The buckling load is, therefore, close to the double modulus load. For relatively slender columns, they are more sensitive to a disturbance of the same magnitude and will bend before the stress reaches the elastic limit, consequently with no stress reversal. In these cases, the buckling load is closer to the tangent modulus load. This effect can be seen clearly in Fig. 8.

Southwell's Method and Its Extension

Referring to Eq. (11), since $P_n = n^2P_{cr}$, we have

$$\delta_n = \bar{\delta}_n/[1 - (P/n^2P_{cr})] \tag{22}$$

As P approaches P_{cr}, we see that

$$\frac{\delta_1}{\bar{\delta}_1} \to \infty, \quad \frac{\delta_2}{\bar{\delta}_2} \to \frac{4}{3}, \quad \frac{\delta_3}{\bar{\delta}_3} \to \frac{9}{8}, \quad \text{etc.}$$

Therefore, $\delta_1 \gg \delta_2 > \delta_3 > \ldots$

By substituting in Eq. (12), it is evident that, if P is a fairly large fraction of P_{cr}, the center deflection δ' is approximately equal to δ_1—i.e.,

$$\delta' \cong \delta_1 = \bar{\delta}_1/[1 - (P/P_{cr})] \tag{23}$$

and Eq. (23) is a close approximation of Eq. (12). The P vs. δ' curve approximates a rectangular hyperbola having the horizontal line $P = P_{cr}$ as an asymptote.

Since the deflections measured in testing are usually referred to the initial position, they are $(\delta' - \bar{\delta})$ rather than δ'. Writing $\delta = \delta' - \bar{\delta}$, Eq. (23) can be rewritten as

$$\delta \cong \delta_1 - \bar{\delta}_1 = \bar{\delta}_1/[(P_{cr}/P) - 1] \tag{24}$$

or

$$P_{cr}(\delta/P) - \delta = \bar{\delta}_1 \tag{25}$$

It is seen that if δ/P is plotted against δ, when P is near to P_{cr}, the testing results will be approximately a straight line, and the inverse of the slope of this line is a measure of the buckling load P_{cr}. Timoshenko[7] also shows that this relationship holds true when there is some eccentricity in applying the load. This is the well-known Southwell's method and has been widely used for estimating the elastic buckling load from testing results. However, because of the happy coincidence that the ultimate load of a column is close to its theoretical elastic buckling load, it is rather a common practice to regard the ultimate load as the buckling load, and the importance of Southwell's method has not been properly emphasized. Actually, Southwell's method represents a much greater achievement than is usually recognized. This is because, for imperfect columns, the buckling load is not defined, and all actual columns are more or less imperfect; thus, in the strict sense, actual testing of buckling is impossible. Southwell's method, however, provides a theoretically sound basis for analyzing the experimental data—from the test results of an imperfect column the buckling load of the corresponding perfect column can be estimated.

In the case of inelastic buckling, Southwell indicated in his paper that his method would not apply, and it is so generally accepted.

Let us refer to Eq. (9). If P is a large portion of the buckling load P_1, in the case of inelastic buckling, it can be replaced by the following relationship:

$$[(d^2y/dx^2) - (d^2y_0/dx^2)] + (P/\overline{EI})y = 0 \tag{26}$$

Assume that $\overline{EI}$ is approximately a constant when P is close to P_{cr}. Then we have

$$\delta' \cong \delta_1 = \bar{\delta}_1/[1 - (P/P_{cr})]$$

and

$$\delta \cong \delta_1 - \bar{\delta}_1 = \bar{\delta}_1/[(P_{cr}/P) - 1] \tag{27}$$

It is thus seen that the Southwell's method can be ex-

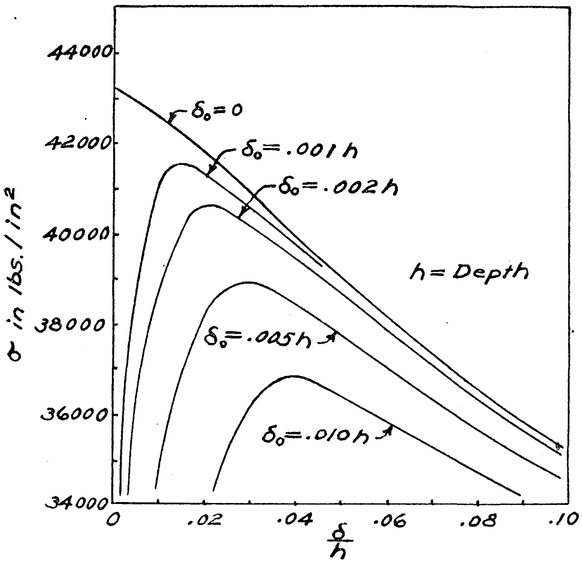

FIG. 9. Calculated load-deflection curves for short columns (l/ρ = 75) with various amounts of initial deflection for mild steel with a yield point = 45,000 lbs. per sq.in. (cf., von Kármán[4]).

tended into the inelastic region, provided that the assumption of $\overline{EI}$ being approximately constant can be justified. As $\overline{EI}$ depends on the mechanical properties of the material, the justification of such an assumption should rely to a great extent on experimental evidence. The simplest way seems to be to apply the conclusion to the experimental results. If the experimental results check with the theoretical prediction, it may be considered that the assumption is valid. This is done by applying the extended Southwell's method to the classical test results of von Kármán,[4] the more recent ones of Gerard, and those of Horsfall and Sandorff [5]* It was found that the resultant plots are good straight lines, thus confirming the validity of the assumption.

Donnell[9] indicated that, even for columns buckled in the plastic range, the first part of the loading is always elastic. By analyzing the measurements in the elastic range according to Southwell's method, the buckling load can therefore be estimated. This argument, however, is not satisfactory because the loading of a column in the elastic range is governed by Eq. (9), and the solution is given by Eqs. (10) and (11), where P_{cr} as obtained by Southwell's method is $C\pi^2EI/L^2$ and not $C\pi^2\overline{EI}/L^2$. Also, if the buckling load is well in the plastic range, the readings in the elastic range may be those that are not the large portions of the buckling load. Although it will be shown later from the test results that the rectangular hyperbolic shape of P vs. δ curves is a good approximation, even well in the elastic range, this agreement can be regarded only as a somewhat fortuitous occurrence.

* When this paper was in preparation, the author received Horsfall and Sandorff's report through the courtesy of F. R. Shanley. It was found that Southwell's method was used in their report, but it was indicated there that the use "is a liberty not justified by theory."

It may be pointed out that, while the presence of initial deflection in the elastic case does not materially influence the ultimate load, the effect of initial deflection in the inelastic case is much more serious. This is because, during the process of loading before the deflection of the column becomes large, the effect of plastic deformation comes in to cause the ultimate load of an actual column to be lower than the theoretical buckling load of the corresponding ideal column. This was shown by von Kármán and is illustrated in Fig. 9. A method for estimating the buckling load from the experimental data in this case is even more important than in the elastic case, since the initial deflection of an actual column is usually difficult to determine, and, consequently, the accuracy of the test result cannot be properly estimated. The limitation of Southwell's method in this case is thus evident. If the initial deflection is too large, the experimental readings may be in the range where the loading curve is not approximately a rectangular hyperbola.

As Southwell originally proposed it, his method requires that the initial deflection reading be taken at zero load. In the vicinity of zero load, deflection readings are usually somewhat questionable. A more general method is suggested by Lundquist,[8] where the initial readings may be taken at any load less than the critical load. Lundquist proved that $(\delta - \delta_1)$ vs. $(\delta - \delta_1)/(P - P_1)$ is also a straight line, where $(\delta - \delta_1)$ is the amount by which the deflections are increased when the axial load on the column is increased from P to P_1.

Instead of measuring the center deflections, it is easier to measure accurately the strains by electric strain gages. If it is assumed that sections remain plane after bending, the curvature $1/R$ of a column at a given cross section is related to the difference in strain at two points, ϵ_1 and ϵ_2, on the particular cross section according to the equation

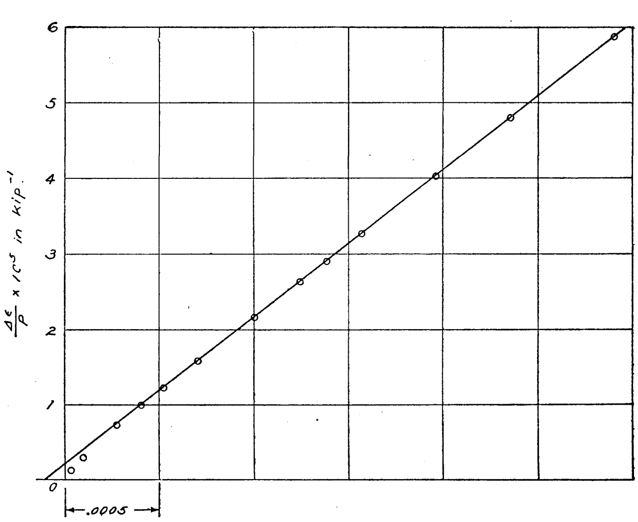

FIG. 10. Modified "Southwell plot" of the test results of Schuette and Roy for a slender column.

TABLE 1

A Comparison Between the Ultimate Load, Euler's Buckling Load, and the Buckling Load Estimated by Southwell's Method for Slender Columns

Column No.	Slenderness Ratio, $\frac{L}{\rho}$	Calculated Buckling Load*	Buckling Load Estimated from Test	Ultimate Load, Kg.	Ratio of Estimated Value to Theoretical Value
1	175.8	3,790	3,710	3,770	0.980
2	146.0	5,475	5,453	5,430	0.995
3a	116.2	8,645	8,590	8,630	0.994
3b	116.1	8,610	8,758	8,750	1.017
4a	103.0	10,980	11,220	11,160	1.022
4b	103.5	10,920	11,090	10,860	1.015
5	95.3	12,780	12,815	12,520	1.003
6	91.3	13,980	13,750	13,580	0.984

* Calculated from Euler's formula by using the actual value of Young's modulus measured by von Kármán, i.e., 2,170,000 kg. per sq.cm.

TABLE 2

A Comparison Between the Ultimate Stress, Theoretical Buckling Stresses, and the Buckling Stress Estimated by Southwell's Method for Short Columns

Column No.	Effective Slenderness Ratio, $(L/\rho)_e$	Calculated Buckling Stress: Tangent modulus theory	Calculated Buckling Stress: Double modulus theory	Buckling Stress Estimated from Test	Ultimate Stress
von Kármán*		Kg. per sq.cm.	Kg. per sq.cm.	Kg. per sq.cm.	Kg. per sq.cm.
7a	88.1	2,400	2,690	2,780	2,760
7b	88.0	2,400	2,690	2,780	2,685
8	82.0	2,600	2,900	2,740	2,740
9a	73.1	2,960	3,050	3,050	3,030
9b	73.1	2,960	3,050	3,105	2,866
10a	58.6	3,100	3,150	3,240	3,185
10b	58.6	3,100	3,150	3,130	3,080
11	53.6	3,120	3,175	3,270	3,165
12a	48.2	3,130	3,210	3,110	3,080
12b	48.2	3,130	3,210	3,050	2,960
13	47.3	3,140	3,215	3,100	3,060
14b	38.2	3,160	3,320	3,480	3,320
15a	28.8	3,220	3,560	3,700	3,395
16	24.8	3,290	4,100	3,900	3,890
17	22.0	3,450	Approx. 4,600	4,500	4,330
Gerard†		Lbs. per sq.in.	Lbs. per sq.in.	Lbs. per sq.in.	Lbs. per sq.in.
1	21.7	41,900	47,500	44,900	44,880
2	21.2	43,000	49,000	48,600	48,150
		Lbs. per sq.in.	Lbs. per sq.in.	Lbs. per sq.in.	Lbs. per sq.in.
Horsfall and Sandorff†	29.9	37,000	42,000	37,600	37,200

* Mild steel columns.
† 24S-T aluminum-alloy columns.

$$1/R = (\epsilon_1 - \epsilon_2)/t \cong (d^2/dx^2)(y - y_0) \qquad (28)$$

where t is the width of the column across which the strain gages are attached. Differentiating Eq. (10) twice and combining it with Eqs. (24) and (28), the difference in strain measurements at the center of the column is

$$\Delta\epsilon = \epsilon_2 - \epsilon_1 = \bar{\delta}_1(\pi/L)^2 t/[(P_1/P) - 1] \qquad (29)$$

when P is near to P_1. It is thus seen that, if $\Delta\epsilon/P$ is plotted against $\Delta\epsilon$, the resultant will also be a straight line. This is confirmed by the test results† of Schuette and Roy[10] in the elastic case (Fig. 10) and by the test results of Gerard in the inelastic case (Fig. 12).

† These results were not given in the original report. The author is indebted to Dr. E. E. Lundquist of N.A.C.A. for his kindness in supplying the test data.

Application of Experimental Results

In order to test the proposed method of analysis, it is necessary to have related test values of load and central deflection for columns that have been loaded as centrally as possible. As was done by Southwell to test his method in the elastic range, von Kármán's test results in the inelastic range are first to be analyzed.

Von Kármán took special precautions to ensure exact centering of his applied loads, and for each column he tabulated his observations of load and deflections during the progress of the test. He classified his columns in three groups—described, respectively, as slender, medium, and thick. Slender columns are those having an L/ρ ratio greater than 90, and the buckling stresses of those are in the elastic range. Medium columns are those having an L/ρ between 45 and 90, corresponding to those having their buckling stress between the proportional limit and yield point. Thick columns are those for which L/ρ is less than 45 and the buckling stresses of which are greater than the yield stress of the material. ρ is the radius of gyration of the cross-sectional area of the column.

The test data of the "slender" columns have been analyzed by Southwell. The buckling loads for the ideal columns thus estimated are listed in Table 1 as compared with the Euler's theoretical buckling loads and the ultimate loads for the actual columns.

The "medium" and "thick" groups of the test data are now to be analyzed. Instead of working with the buckling load, it was felt that working with the buckling stress was more convenient in these cases. Eq. (24) can be replaced by the following expression:

$$\delta = \bar{\delta}_1/[(\sigma_{cr}/\sigma) - 1] \tag{30}$$

where $\sigma = P/A$ and $\sigma_{cr} = P_{cr}/A$. It is assumed that the cross-sectional area of the column remains constant with an increase of load. δ/σ is plotted against δ in Figs. 11a and 11b. Except for columns 14a and 15b, where the recorded data are too few for such an analysis, it is seen that the test data lie closely on the corresponding straight lines, thus confirming the applicability of the method. The buckling stresses so determined are listed in Table 2. The buckling stresses calculated by the tangent modulus formula and the double modulus formula, as well as the ultimate stresses of the actual columns, are also tabulated in Table 2 for comparison. The buckling stresses as computed by the double modulus formula were computed by von Kármán. The computation of the buckling stresses by the tangent formula is based on the average tangent modulus given by von Kármán. It is seen that some of the buckling stresses so estimated are greater than the corresponding double modulus stresses. This is probably because the moduli used in the computation are the average values rather than the actual values for the particular specimens. A typical stress-strain curve tested by Gerard is shown in Fig. 13, and the variations of the moduli of a particular specimen to the average values can be large.

In order to test the proposed procedure still further, Gerard, formerly of the Republic Aviation Corporation, has kindly supplied the author with the test data of two 24S-T aluminum-alloy columns of rectangular cross sections (approximately $1^1/_4$ by $^1/_2$ in.). The columns were designed to fail in the plastic range and were manufactured with great care to ensure as little initial curvature as possible. The specimens were tested flat-ended and were equipped with three electric resistance-type wire strain gages opposite each other on the wider sides, one pair being at the mid-section and the other two near the estimated inflection points. Referring to Eqs. (12) and (23), it is seen that, as load is increased, the column will bend approximately into a sine curve, because the first harmonic becomes large and

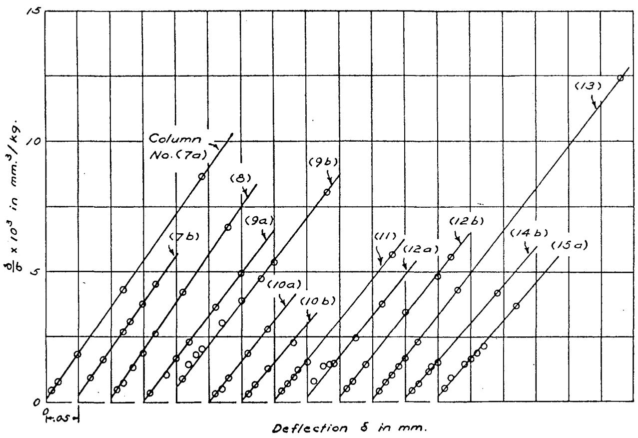

FIG. 11a. "Southwell plot" of von Kármán's test results for short columns.

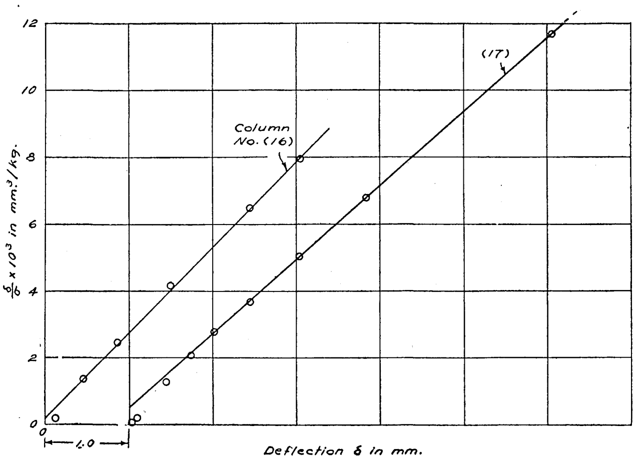

FIG. 11b. "Southwell plot" of von Kármán's test results for short columns.

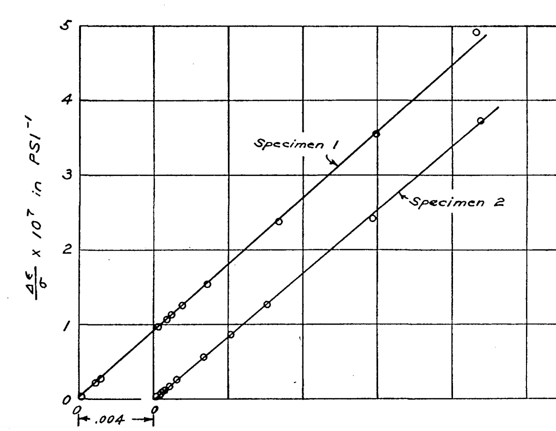

FIG. 12. "Southwell plot" of Gerard's test results for short columns.

other harmonics are little magnified by the load. Points of zero curvature can be approximately determined by passing a sine curve through $\Delta\epsilon$ readings opposite each other along the length of the column. The effective length of the column is then the distance between these points of zero curvature or the inflection points. The test data are plotted in Fig. 12. It is seen that the test

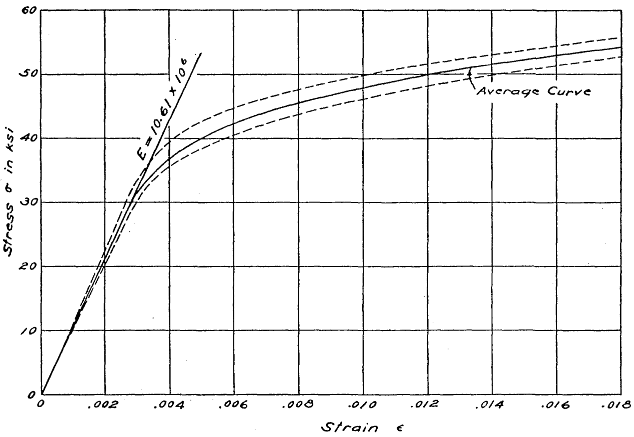

FIG. 13. Compressive stress-strain curve of 24S-T aluminum alloy as tested by Gerard.

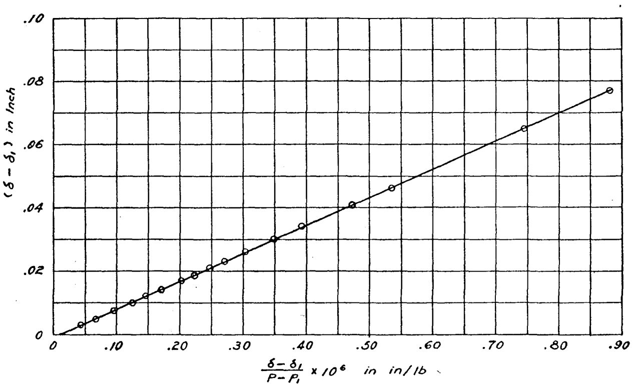

FIG. 14. Lundquist's modified form of "Southwell plot" of the test results of Horsfall and Sandorff for a short column.[5]

points lie closely on the respective straight lines. The buckling stresses so determined are listed in Table 2, together with the theoretical values and the ultimate stresses.

Another set of accurately measured data is that of Horsfall and Sandorff,[5] of Lockheed Aircraft Corporation. Their data are plotted according to Southwell's method modified by Lundquist. The result is reproduced here as Fig. 14 and offers further evidence to the validity of the method. The test procedures are described in detail in references 2 and 5.

It is interesting to note from the results in Table 2 that all the buckling loads estimated by Southwell's method are either equal to, or greater than, the corresponding ultimate loads, as they should be in these cases.

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