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Inelastic Column Theory

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SUMMARY

The action of a column in the plastic range is analyzed on the basis that bending may proceed simultaneously with increasing axial load. This leads to a new column formula that includes both the tangent-modulus (Engesser) and the reduced-modulus (von Kármán) formulas. It is shown that bending begins at the tangent-modulus load and that the column load increases with increasing lateral deflection, approaching the reduced-modulus load as a limit if the tangent modulus is assumed to remain constant.

INTRODUCTION

In a recently published paper,[1] the author stated that the reduced-modulus (or double-modulus) theory is not correct for predicting the maximum load up to which a perfect column will remain straight. This is because it is possible for the column to bend simultaneously with increasing axial load. Under such conditions it is possible to have bending without introducing any strain reversal, upon which the reduced-modulus theory depends. In reference 1 it was stated that the column will begin to bend as soon as the axial load exceeds the value predicted by the tangent-modulus (Engesser) theory. It appeared likely that the Engesser load could be exceeded but that the reduced-modulus load could not be reached. In this paper it will be shown that for an idealized simplified column, this is actually true.

The three basic column formulas may be written as follows (assuming pin ends and zero eccentricity):

$$\text{(Euler)} \quad P_e = \pi^2 EI/L^2 \tag{1}$$

$$\text{(Engesser)} \quad P_t = \pi^2 E_t I/L^2 \tag{2}$$

$$\text{(reduced modulus)} \quad P_r = \pi^2 E_r I/L^2 \tag{3}$$

where

P = critical load
I = moment of inertia of column cross section
L = column length
E = Young's modulus (slope of stress-strain diagram in elastic range)
E_t = tangent modulus (local slope of stress-strain diagram in inelastic range)
E_r = reduced modulus (an effective value lower than E and higher than E_t)

Eqs. (1), (2), and (3) differ only in the value used for the effective modulus of elasticity. Since the Euler equation applies only in the elastic range, the problem of column action in the inelastic (plastic) range centers around Eqs. (2) and (3). An excellent summary of the history of these two theories is given in reference 2, from which the following is quoted:

"What is here called the double-modulus theory has frequently been called also the Considère-Engesser theory and Kármán's theory. Many competent engineers are mistaken as to the origin of the theory, and a brief account of its development will not be out of place. In 1891 there was published a memoir included as an appendix (annexe) to the proceedings of the Congrès International des Procédés de Constructions, held in Paris from the 9th to the 14th of September, 1889, in which A. Considère pointed out that, as an ideal column stressed beyond the proportional limit begins to bend, the stress on the concave side increases according to the law of the compressive stress-strain diagram, and the stress on the convex side decreases according to Hooke's law, and that therefore the strength would be given by

$$P = \pi^2 \bar{E} I/L^2$$

in which $\bar{E}$ is a modulus, the value of which lies between the modulus of elasticity and the tangent modulus. Considère realized that $\bar{E}$ was a function of P/A, the average stress in the column, but did not go further than to point out this fact.

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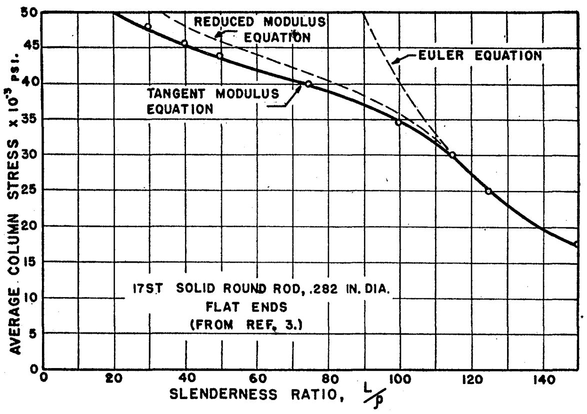

FIG. 1. Column theories and test data (Alcoa).

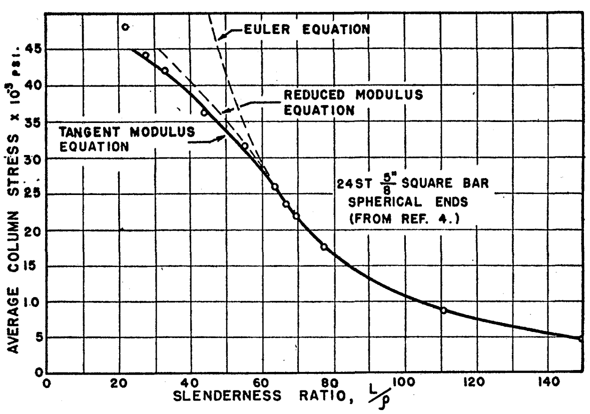

FIG. 2. Column theories and test data (Van den Broek).

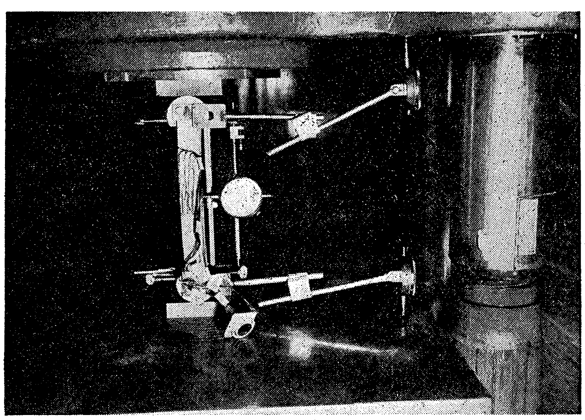

FIG. 3. Pin-end column test arrangement.

"Earlier in 1889 Fr. Engesser presented his tangent-modulus theory, and in 1895 Félix Jasinski ('Noch ein Wort zu den Knickfragen,' *Schweizerische Bauzeitung*, Vol. XXV, No. 25, p. 172, June 22, 1895) pointed out that this theory was not correct and called attention to Considère's work. He stated that at that time it was impossible, however, to determine theoretically the form of the function $\bar{E}$. Thereupon Engesser ('Ueber Knickfragen,' *Schweizerische Bauzeitung*, Vol. XXVI, No. 4, p. 24, July 27, 1895) acknowledged the error in his original theory and replied that the possibility of determining $\bar{E}$ theoretically was in no wise out of the question, and he determined it in the general form. . . . . Nothing further was done apparently until Kármán ('Untersuchungen über Knickfestigkeit,' *Forschungsarbeiten*, Nr. 81, 1910) presented the theory again, added the actual evaluation of $\bar{E}$ for the rectangular cross section and the idealized *H*-section (consisting of infinitely thin flanges and negligible web), and gave the theory new life by making a series of careful tests designed to afford a check on the theory. Since then $\bar{E}$ has been evaluated for other cross sections by a number of writers."

Although the reduced-modulus (or double-modulus) theory has long been accepted as the exact theory of column action, the simpler tangent-modulus theory has been found to be much easier to use, since it is not affected by the shape of the cross section. Since it gives lower critical loads than the reduced-modulus theory, it is also preferred by engineers on the basis of safety. Finally, test data indicate that the actual buckling loads are usually closer to the tangent-modulus values than to the reduced-modulus values. A typical example is shown in Fig. 1, which is reproduced from reference 3 (tests by the Aluminum Research Laboratories). Reference 4 also shows test data that agree with the Engesser theory. Fig. 2 is based on Fig. 4 of this reference, which includes a number of tests made by Professor Van den Broek.

TEST DATA

In an effort to obtain a clue, the author had a simple test performed by the Lockheed Research Laboratory. A pin-ended 24ST aluminum-alloy column of rectangular cross section (2 by $1^1/_4$ in.) was designed to fail in the plastic range and was equipped with electric strain gages on opposite sides, at the mid-section (Fig. 3). (Details of the test are omitted here but are reported in reference 5.)

The results obtained are plotted in two different ways in Figs. 4 and 5.

Fig. 4 shows that the strain distribution remained substantially constant up to a load of about 40,000 lbs. and then gradually shifted, indicating bending. Up to a load of about 87,000 lbs. there was no strain reversal. Further loading caused strain reversal to take place. At the maximum load of 92,500 lbs., considerable strain reversal had taken place and the axis of rotation of the cross section appeared to be at about one-third of the width of the column.

Fig. 5 shows the strain on opposite sides of the column plotted against load. The slight divergence corresponds to an initial eccentricity of about 0.001 in. If the column were perfect and remained straight up to the reduced-modulus load, the two curves would coincide and would be represented by the dashed line OA. (The upward curvature reflects the shape of the compression stress-strain diagram.) Point B

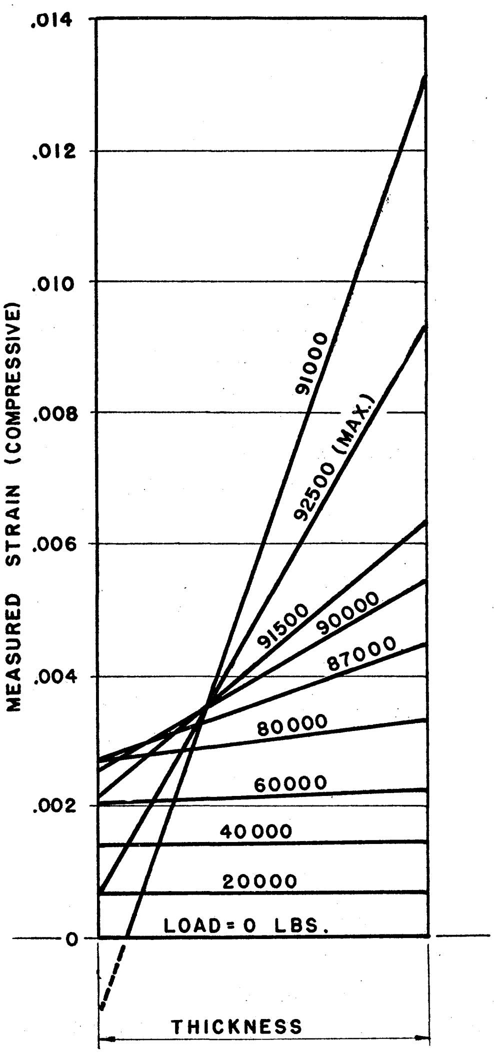

FIG. 4. Strain distribution as determined in a column test.

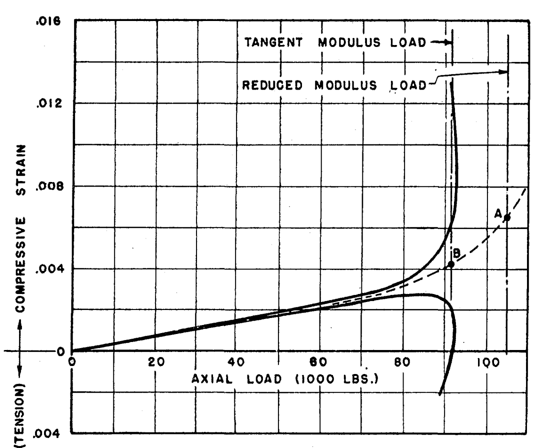

FIG. 5. Strain on opposite faces of column from test data.

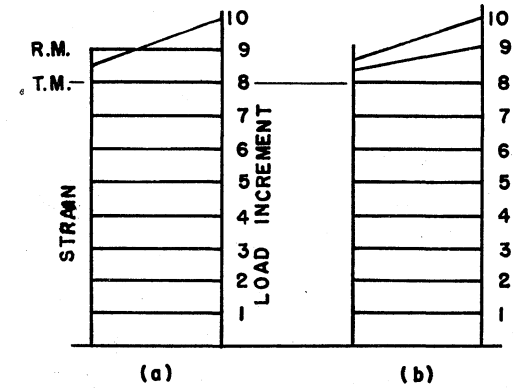

FIG. 6. Alternative types of strain distribution across column cross section.

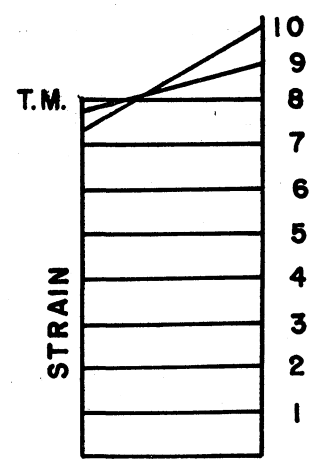

FIG. 7. Type of strain distribution needed to permit loading beyond tangent-modulus load.

represents the upper limit as predicted by the tangent-modulus theory.

Fig. 5 shows that if the column were to remain straight up to the reduced-modulus load there could be no strain reversal below that load. What, then, can supply the extra effective value of E needed to prevent buckling beyond the tangent-modulus load? The obvious answer is that the column cannot remain straight beyond the tangent-modulus load; there must be a definite amount of strain reversal as soon as the load is further increased. This should cause the two curves to separate at point B, one starting downward and the other upward.

It can now be seen that in the derivation of the reduced-modulus theory a questionable assumption was made. It was assumed, by implication at least, that the column remains straight while the axial load is increased to the predicted critical value, *after which* the column bends, or tries to bend. Actually, the column is free to "try to bend" at any time. There is nothing to prevent it from bending *simultaneously* with increasing axial load. Under such a condition it is possible to obtain a nonuniform strain distribution without any strain reversal taking place. The difference between the two assumptions is shown diagrammatically in Fig. 6.

Fig. 6b, however, still represents a paradox. There is no strain reversal indicated; hence, the value of E_t must apply over the entire cross section; therefore, the column load cannot exceed the tangent-modulus

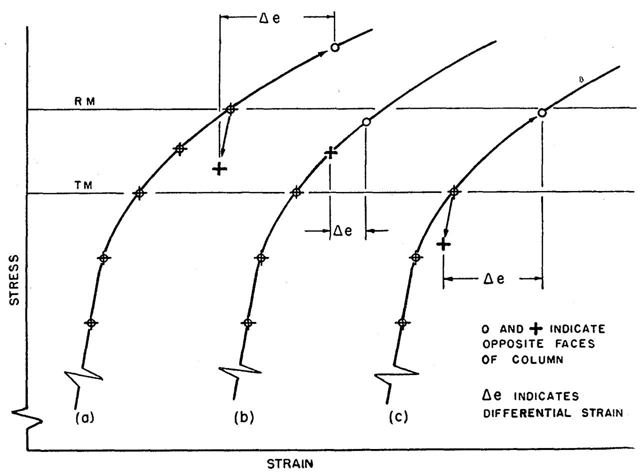

FIG. 8. Alternative types of strain progression.

value. If the load is to go any higher, some strain reversal must begin at the tangent-modulus load. The picture might then look something like Fig. 7, in which each succeeding increment of loading beyond the tangent-modulus load causes some additional strain reversal. The fact that this picture resembles the actual distribution shown in Fig. 4 is significant. (NOTE: These statements do not necessarily apply to the entire *length* of the column, since the ratio of bending moment to axial load varies from end to end.)

Another way to visualize the two different theories is shown in Fig. 8, in which the points represent conditions on opposite edges of the column. In the reduced-modulus theory it is assumed that the points stay together until the critical load is reached (Fig. 8a). Fig. 8b shows that they may separate without involving any strain reversal. Since the average stress in Fig. 8b is greater than that corresponding to the tangent-modulus theory, it represents an impossible condition. Fig. 8c shows what would have to happen if the average stress were to exceed the tangent-modulus value.

On the basis of the foregoing reasoning the author predicted, in reference 1, that (a) bending will begin as soon as the tangent-modulus load is exceeded; (b) the maximum column load will be reached somewhere between the loads predicted by the two theories.

MATHEMATICAL ANALYSIS

In order to prove the last statement, for at least one case, the problem has been greatly simplified by adopting a suggestion made by E. I. Ryder, of the Civil Aeronautics Authority. This consists in working with a two-legged hinged column in which the hinge consists of a unit "cell" formed from two small axial elements. As shown in Fig. 9, the two legs of the column are assumed to be infinitely rigid. If the dimensions of the unit cell are sufficiently small with respect to the column length, L, it can be assumed that there is a simple hinge action about the center of the cell. This device reduces the problem to elementary form by eliminating the mathematical work involved in integrating over the cross section and over the length of the column.

As shown in Fig. 9, the two elements of the column cell are assumed to have deflected in opposite directions through the distances e_1 and e_2, which may be regarded as the strains that occur after the column starts to bend. If e_1 and e_2 are assumed to be equal and opposite in sense (as shown in Fig. 9), pure bending is indicated. This restriction will *not* be applied, however; e_1 and e_2 may have different values, indicating combined bending and variation in axial load.

The critical load for such a column is easily determined by equating external and internal bending moments. The lateral deflection, d, is first expressed in terms of strain:

$$d = \frac{\alpha L}{2} = \frac{1}{2}\left(\frac{e_1}{2} + \frac{e_2}{2}\right)L = \frac{L}{4}(e_1 + e_2) \qquad (4)$$

The external bending moment at the hinge is

$$M_e = Pd = (PL/4)(e_1 + e_2) \qquad (5)$$

The axial load in each flange element, due to bending, is

$$\left.\begin{aligned} P_1 &= e_1E_1(A/2) \\ P_2 &= e_2E_2(A/2) \end{aligned}\right\} \qquad (6)$$

Note that E_1 and E_2 indicate the value of E which is considered to be effective for each flange element.

The internal bending moment (about the hinge point) may be expressed as

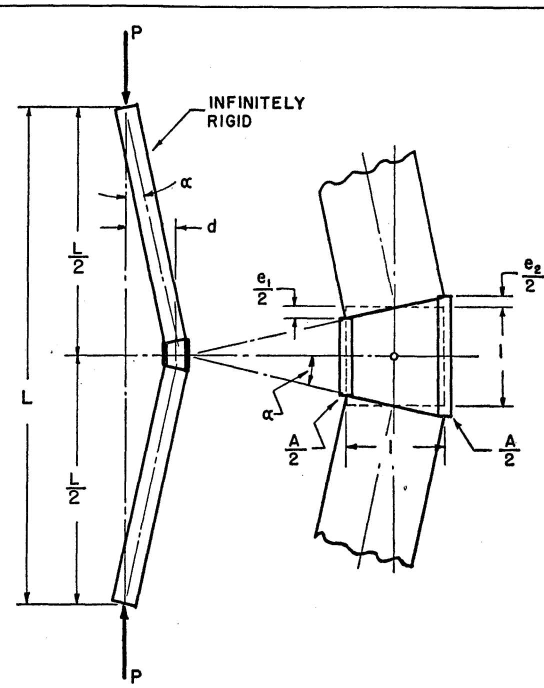

FIG. 9. Simplified two-flange column.

$$M_i = \frac{1}{2}P_1 + \frac{1}{2}P_2 = \frac{1}{2}e_1E_1\frac{A}{2} + \frac{1}{2}e_2E_2\frac{A}{2} = \frac{A}{4}(e_1E_1 + e_2E_2) \quad (7)$$

Equating internal and external bending moments [Eqs. (5) and (7)],

$$(PL/4)(e_1 + e_2) = (A/4)(e_1E_1 + e_2E_2)$$

from which

$$P = (A/L)[(e_1E_1 + e_2E_2)/(e_1 + e_2)] \quad (8)$$

Now in either the Euler or Engesser equations $E_1 = E_2$. Eq. (8) therefore reduces to the following two equations:

$$\text{(Euler)} \qquad P_e = AE/L \quad (9)$$

$$\text{(Engesser)} \quad P_t = AE_t/L \quad (10)$$

If it is now assumed that element (1) undergoes increasing compressive stress while element (2) has a decreasing compressive stress, E_1 and E_2 may be replaced by E_t and E, respectively.

Let $k = E/E_t$. Then

$$E_1 = E_t$$

$$E_2 = kE_t$$

Substituting in Eq. (8),

$$P = (AE_t/L)[(e_1 + ke_2)/(e_1 + e_2)] \quad (11)$$

This is the same as the Engesser formula except for the added term, which will now be further examined.

From Eq. (4)

$$e_1 + e_2 = 4d/L$$

and

$$e_1 = 4d/L - e_2 \quad (12)$$

Substituting these values in Eq. (11),

$$P = \frac{AE_t}{L}\frac{L\,[(4d/L) - e_2 + ke_2)]}{4d}$$

$$P = \frac{AE_t}{L}\left[1 + \frac{L}{4d}(k-1)e_2\right] \quad (13)$$

Another expression for P may be obtained by assuming that, after the tangent-modulus load is reached, the column load continues to increase. This increase is given by the difference between the element loads P_1 and P_2, which can be expressed, from Eq. (6), as

$$\Delta P = P_1 - P_2 = e_1E_1(A/2) - e_2E_2(A/2)$$

Substituting for E_1 and E_2,

$$\Delta P = (A/2)E_t(e_1 - ke_2)$$

Substituting for e_1,

$$\Delta P = \left(\frac{A}{2}\right)E_t\left(\frac{4d}{L} - e_2 - ke_2\right) = \frac{A}{2}E_t\left[\frac{4d}{L} - (1+k)e_2\right] \quad (14)$$

This value should be added to the tangent-modulus load to obtain the total value for P.

$$P = P_t + \Delta P = \frac{AE_t}{L} + \frac{A}{2}E_t\left[\frac{4d}{L} - (1+k)e_2\right]$$

$$P = \frac{AE_t}{L}\left\{1 + \left[2d - \frac{L}{2}(1+k)e_2\right]\right\} \quad (15)$$

From Eqs. (13) and (15) the following equation can be set up:

$$\frac{L}{4d}(k-1)e_2 = 2d - \frac{L}{2}(1+k)e_2 \quad (16)$$

From this,

$$e_2 = 8d^2/L[k - 1 + 2d(1+k)] \quad (17)$$

Substituting this value of e_2 in Eq. (13),

$$P = \frac{AE_t}{L}\left[1 + \frac{2d(k-1)}{k - 1 + 2d(1+k)}\right] \quad (18)$$

This may be reduced to

$$P = \frac{AE_t}{L}\left(1 + \left\{1\bigg/\left[\frac{1}{2d} + \left(\frac{k+1}{k-1}\right)\right]\right\}\right) \quad (19)$$

If the ratio E_t/E is represented by the usual symbol τ, Eq. (19) becomes

$$P = \frac{AE_t}{L}\left(1 + \left\{1\bigg/\left[\frac{1}{2d} + \left(\frac{1+\tau}{1-\tau}\right)\right]\right\}\right) \quad (19a)$$

Let

$$R = P/P_t \quad (20)$$

Then, from Eq. (10),

$$R = 1 + \left\{1\bigg/\left[\frac{1}{2d} + \left(\frac{k+1}{k-1}\right)\right]\right\} \quad (21)$$

or

$$R = 1 + \left\{1\bigg/\left[\frac{1}{2d} + \left(\frac{1+\tau}{1-\tau}\right)\right]\right\} \quad (21a)$$

In Fig. 10, R is plotted against lateral deflection for two different values of τ. [NOTE: Eqs. (18) to (21a) do not apply when $E_t = 0$, since the limiting column load is then determined by the stress at which this occurs.]

It should be noted that Fig. 10 corresponds to the usual plot of column load against lateral deflection. If the tangent modulus had been assumed to decrease with increasing stress (as it usually does), the curves would rise to some maximum value and then start downward again. It is interesting to note that in the very short column range, where the tangent modulus approaches a constant value, test points often lie closer to the reduced-modulus curve than to the tangent-modulus curve.

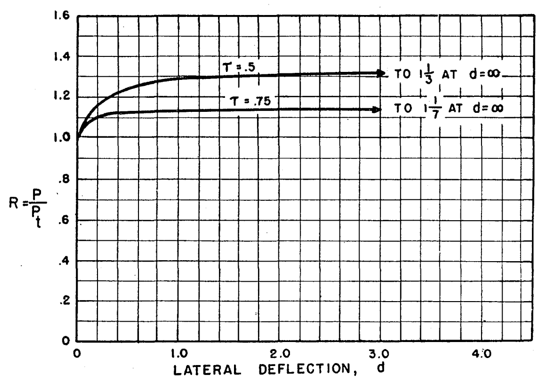

FIG. 10. Variation of column load with lateral deflection, assuming E_t constant.

Comparison with Reduced-Modulus Theory

The equation for the critical column load will now be derived on the basis of the assumptions originally used in the reduced-modulus theory. It will be assumed that the column remains straight up to the critical load P_r, after which it bends. The derivation proceeds as before, up to and including Eq. (13). Now, instead of assuming that there can be an increase in load, ΔP, it will be assumed that $\Delta P = 0$. Then $P_1 = P_2$ and $E_t e_1 = E e_2$, from which

$$e_2 = e_1(E_t/E) = e_1/k \tag{22}$$

Substituting for e_1 from Eq. (12),

$$e_2 = (1/k)[(4d/L) - e_2]$$

$$e_2 = (4d/L)[1/(1 + k)] \tag{23}$$

Substitution of this value in Eq. (13) yields

$$P_r = (AE_t/L)\{1 + [(k - 1)/(k + 1)]\} \tag{24}$$

Eq. (24) obviously represents the limiting value of Eq. (19) as d approaches infinity. This proves that, for the simplified column, the reduced-modulus theory gives the limiting value for the column load as the lateral deflection approaches infinity, assuming that the value of the tangent modulus, E_t, remains constant.

The actual upper limit for the column load will depend largely on the manner in which E_t varies with increasing strain, as shown by von Kármán.[6] It is important to note, however, that, even for a perfect column, lateral deflection must take place as soon as the tangent-modulus load is exceeded and that the load predicted by the reduced-modulus theory can never be reached, even if there is no dropping off in the tangent modulus with increasing strain.

Eq. (19) represents, for the simplified column and for a constant value of E_t, the complete theory of column action. It includes the Euler equation (when $k = 1$) and the Engesser (tangent-modulus) equation (when $d = 0$) and approaches the reduced-modulus equation as a limit (as $d \to \infty$). It is suggested that an equation of this type be called the *column equilibrium equation* to avoid any question as to the definition of buckling or instability.

Variation of Strain with Load

The equation already derived may be used to find equations for the strains in the two column elements. From Eqs. (21), (12), and (17) the following equations may be derived:

$$e_1 = \frac{2}{L}\left[\frac{k - R}{\left(\dfrac{k-1}{R-1}\right) - (k + 1)}\right] \tag{25}$$

$$e_2 = \frac{2}{L}\left[\frac{R - 1}{\left(\dfrac{k-1}{R-1}\right) - (k + 1)}\right] \tag{26}$$

From these equations it should be possible to obtain a graphical picture of the variation of strain against column load, to compare with the experimental curve of Fig. 5. Since Eqs. (25) and (26) apply only after the column has started to bend, at the load P_t, it is necessary to determine the average compressive strain at the load P_t. The stress, obtained from Eq. (10), is

$$P_t/A = E_t/L$$

The actual strain corresponding to this stress would be obtained from the stress-strain diagram of the material. If the stress were in the elastic range, the strain would be

$$e_t = P/AE = E_t/EL = 1/kL \tag{27}$$

This would be the value of the strain at $R = 1$. (For most materials the strain would actually be considerably higher, but this does not affect the general results.) The additional strain in each element, beyond $R = 1$, will be put in terms of the strain e_t, giving

$$\frac{\Delta e_1}{e_t} = 2\left[\frac{k(k - R)}{\left(\dfrac{k-1}{R-1}\right) - (k + 1)}\right] \tag{28}$$

$$\frac{\Delta e_2}{e_t} = 2\left[\frac{k(R - 1)}{\left(\dfrac{k-1}{R-1}\right) - (k + 1)}\right] \tag{29}$$

These values are plotted in Fig. 11, which represents a typical variation of strain with column load, for a value of $k = 1.333$.

Although Fig. 11 is based on extreme simplification of the problem and does not give a true picture of typical conditions, it has a general resemblance to Fig. 5, which was obtained from actual test data. It is interesting to note that on the concave side of the column the compressive strain increases rapidly after the tangent-modulus load is reached, while on the convex side the strain starts to decrease rather slowly.

The rapid increase of compressive strain which is required to obtain additional load beyond P_t will usually involve a large reduction in E_t. This indicates that in most actual cases the maximum column load will exceed the tangent-modulus load by only a slight amount.

Conclusions

Although the foregoing analysis is based on a hypothetical column that bears little resemblance to an actual column, extension of the theory to the more general case is largely a matter of mathematics. The fundamental principles of column action in the inelastic range will not be changed by such generalization. The following conclusions can therefore be drawn from the analysis of the simplified column:

(1) The tangent-modulus (Engesser) formula gives the maximum load at which an initially straight, centrally loaded column will remain straight.

(2) The column load may exceed the tangent-modulus load but cannot be greater than the reduced-modulus load. (The latter statement has not been proved for the general case.)

(3) Loading beyond the tangent-modulus load will cause bowing, which will produce permanent bending deformation (eccentricity).

(4) There will be some portion of the column cross section for which the stress will never exceed the tangent-modulus stress.

(5) After the tangent-modulus load is exceeded, the compressive strain over a portion of the cross section will increase much more rapidly than the average strain.

(6) For most engineering materials the decrease in tangent modulus with increasing strain will limit the amount by which the column load may exceed the tangent-modulus value.

(7) The idealized simplified column treated in this paper cannot sustain the reduced-modulus load unless the column is laterally supported while the axial load is being applied.

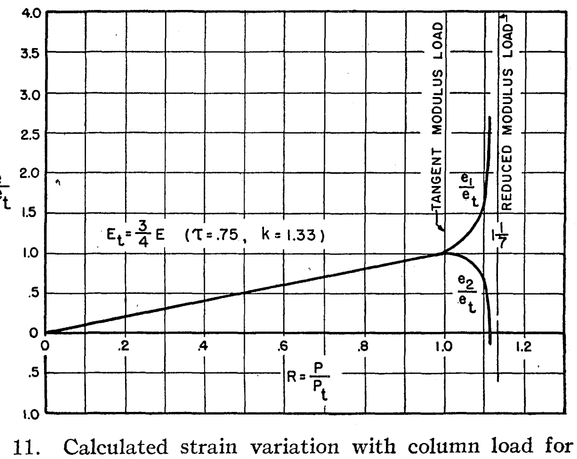

FIG. 11. Calculated strain variation with column load for a hypothetical column.

(8) When supported against bowing during axial loading, a column in the inelastic range will sustain a greater axial load (with support removed) than when loaded without such support.

(9) The tangent-modulus (Engesser) equation should be used as a basis for determining the buckling strength of members in the inelastic range.

In conclusion, it would seem fitting to repeat a statement made by von Kármán[6] to the effect that "Engesser appears to have been the first to recognize properly the nature of 'inelastic buckling.'" It is interesting that aircraft engineers, in seeking greater accuracy in the inelastic range, have gone back to the formula that was first suggested by Engesser over 50 years ago.

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Discussion

Dr. Theodore von Kármán: "The theory of buckling of a column subjected to axial load beyond the limit of elasticity has been discussed by many authors in the last three decades. Most of the objections, however, were not valid. Now Mr. Shanley comes along with an objection that is worthy of consideration.

"Both Engesser's and my own analyses of the problem were based on the assumption that the equilibrium of the straight column becomes unstable when there are equilibrium positions infinitesimally near to the straight equilibrium position under the same axial load. The correct answer to this question is given by replacing, in Euler's equation, Young's modulus by the so-called reduced modulus. Mr. Shanley's analysis represents a generalization of the question. His procedure can be formulated as follows: What is the smallest value of the axial load at which a bifurcation of the equilibrium positions can occur, regardless of whether or not the transition to the bent position requires an increase of the axial load? The answer to this question is that the first equilibrium bifurcation from the straight equilibrium configuration occurs at a load given by the Euler formula when the Young's modulus is replaced by the tangent modulus. In fact, one can construct sequences of equilbrium positions starting from any load between the two limiting values corresponding to the tangent and the reduced moduli. The inclination of the equilibrium lines representing the load as a function of the deflection is steepest for the line starting from the lower limiting load and becomes zero for the line starting from the upper limiting load. The equilibrium lines have an envelope that starts from the lower limiting load and—at least as long as the stress-strain curve can be considered straight and the deflection small—approaches asymptotically the load computed with the reduced modulus.

"Two aspects of the question are worthy of mention:

"(a) My original analysis, and also Engesser's, is a generalization of the reasoning used in the theory of elastic buckling. Why does this not cover all possible equilibrium positions in the inelastic case? Obviously, it is not because of the nonlinearity of the stress-strain relation in the inelastic range but because of the nonreversible character of the process. There are infinite values of permanent strain which may correspond to the same stress, corresponding to different history of the loading and unloading procedure. Hence, the definition of the stability limit must be revised for nonreversible processes. This necessity was intuitively recognized by Mr. Shanley, which is, I believe, a great merit of his paper.

"(b) Although the Euler formula with the tangent modulus does not, in general, give the maximum axial load to which the column can be subjected without large deflection, it is conservative and therefore advisable to use this formula for practical computation of column loads. As I have shown in my paper of 1909, also the load deflection curve that starts from the upper limiting load in general soon assumes a negative slope. Consequently, it is difficult to determine the actual peak of the axial load. It will certainly be between the two values that correspond to the tangent and the reduced moduli. These two values can be respectively designated as the lower and upper limits of the critical load."