The Column Paradox

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Since this article was received, the author has informed us that he has worked out the equilibrium equation for a simplified, two-flange column. The column starts to bend at the tangent modulus load and the load increases with increasing lateral deflection. If the tangent modulus is assumed to be constant, the column load is asymptotic to the value predicted by the reduced modulus theory. This development will be covered by a second paper, now in preparation.—EDITOR.

Introduction

As shown by von Kármán in his classical paper,[1] column action in the inelastic range (beyond the proportional limit) involves two different moduli of elasticity: the elastic (Young's) modulus, and the tangent modulus (local slope of stress-strain diagram). The "reduced modulus" is a function of both of these and is also affected by the shape of the cross section. By using the reduced modulus in place of Young's modulus, Euler's equation has been generalized to include stresses well beyond the elastic limit.

Reduced Modulus Theory

If the tangent modulus is used directly in the Euler formula, the resulting critical load is somewhat lower than that given by the reduced modulus theory. This simpler formula, originally proposed by Engesser, is now widely used by engineers, since it gives values that agree very well with test data. Nevertheless, the reduced modulus formula is still generally considered to be the true theoretical solution for perfect columns, and the lower test values are explained by references to unavoidable eccentricities, testing technique, and other errors.

The reduced modulus theory is derived by assuming that, *after* the column reaches the critical uniform stress, the column bends. This causes the strain to decrease on one side and increase on the other. For the increasing strains the resulting stresses are given by the tangent modulus, while for the decreasing strains the elastic modulus gives the relation between strain and stress. Tests have shown that, *when the strain reverses*, actual conditions are in accordance with this theory.

Discussion

But there is an implied assumption in the derivation of the reduced modulus theory that is open to question. It is, in effect, assumed that something keeps the column straight while the strain increases from that predicted by the tangent modulus theory to the higher value derived from the reduced modulus theory. Actually, there is nothing (except the column's bending stiffness) to prevent the column from bending *simultaneously* with increasing axial loading. Under such conditions the compressive strain could increase on one edge of the column while remaining constant on the other, or it could increase at a different rate on opposite edges. If such action were assumed, the tangent modulus would apply over the entire cross section, and the theoretical buckling load would be that predicted by the tangent modulus theory. This creates a paradox, because, if all of the strains equal or exceed the tangent modulus value, the average strain will be *greater* than that predicted by the tangent modulus theory.

The assumptions involved in the reduced modulus theory also represent a paradox. The theory predicts that the column will remain straight up to the calculated maximum load, but it also shows that some strain reversal is needed in order to provide the additional column stiffness required beyond the tangent modulus load. It is impossible to have strain reversal in a straight column.

Evidently, the maximum column load will be reached somewhere between the loads predicted by the two theories. The entire problem should be reviewed on the basis that axial loading and bending can occur *simultaneously*. The use of the principle of superposition is not valid in this case.

Conclusions

Since it will require some reversal of strain in order to exceed the tangent modulus load, it can be predicted that bending will begin as soon as that load is exceeded. The tangent modulus theory can therefore be regarded as correct for predicting the maximum axial load at which a perfect centrally loaded column will remain straight.

To find the maximum load for a perfect column in the plastic range, it will evidently be necessary to analyze each case separately, using a step-by-step process and employing the general methods followed by von Kármán in his solution of the eccentric column.

Reference

[1] Forschungsarbeiten, No. 81, 1910, Berlin; see any modern textbook on advanced structures.

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